Advanced Systems Lab

Spring 2025

Lecture: Discrete Fourier transform, fast Fourier transforms

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Linear Transforms

Overview: Transforms and algorithms

Discrete Fourier transform

Fast Fourier transform algorithms (FFTs)

After that:

- *Optimized implementation and autotuning (FFTW)*
- *Automatic program synthesis (Spiral)*

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FFT References

FFTs:

- *Cooley and Tukey, **An algorithm for the machine calculation of complex Fourier series**,” Math. of Computation, vol. 19, pp. 297–301, 1965*
- *Nussbaumer, **Fast Fourier Transform and Convolution Algorithms**, 2nd ed., Springer, 1982*
- *van Loan, **Computational Frameworks for the Fast Fourier Transform**, SIAM, 1992*
- *Tolimieri, An, Lu, **Algorithms for Discrete Fourier Transforms and Convolution**, Springer, 2nd edition, 1997*
- *Franchetti, Püschel, Voronenko, Chellappa and Moura, **Discrete Fourier Transform on Multicore**, IEEE Signal Processing Magazine, special issue on ``Signal Processing on Platforms with Multiple Cores'', Vol. 26, No. 6, pp. 90-102, 2009*

Complexity: *Bürgisser, Clausen, Shokrollahi, Algebraic Complexity Theory, Springer, 1997*

History: *Heideman, Johnson, Burrus: Gauss and the History of the Fast Fourier Transform, Arch. Hist. Sc. 34(3) 1985*

Linear Transforms

Very important class of functions: signal processing, communication, scientific computing, …

Mathematically:
Change of basis = Multiplication by a fixed (entries are constants) matrix T

$$\begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} = y = Tx \longleftarrow \boxed{T\cdot} \longleftarrow x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

$$T = [t_{k,\ell}]_{0 \le k,\ell < n}$$

Output ***Input***

Equivalent definition: Summation form

$$y_k = \sum_{\ell=0}^{n-1} t_{k,\ell} x_\ell, \quad 0 \le k < n$$

Operations in linear transforms: additions and multiplications by constants

Linear Transforms

Compute: y = Tx   x: input vector, y: output vector, T: fixed transform matrix

Example: Discrete Fourier transform (DFT)

1. form (standard in signal processing):

given: $x_0, \ldots, x_{n-1}$

compute:
$$y_k = \sum_{\ell=0}^{n-1} e^{-2k\ell\pi i/n} x_\ell, \quad k = 0, \ldots, n-1$$
$$= \sum_{\ell=0}^{n-1} \omega_n^{k\ell} x_\ell, \quad k = 0, \ldots, n-1, \quad \omega_n = e^{-2\pi i/n}$$

primitive nth root of 1

2. form (we will use):

given: $(x_0, \ldots, x_{n-1})^T$

compute: $y = \mathbf{DFT}_n \cdot x, \quad \mathbf{DFT}_n = [\omega_n^{k\ell}]_{0 \le k,\ell < n}$

How does the DFT_2 matrix look?
Second row of DFT_4 matrix?

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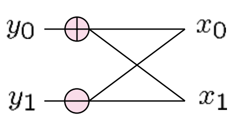
Smallest Relevant Example: DFT, Size 2

Transform (matrix): $\mathbf{DFT}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

How many ops to compute the DFT_2 of a vector?

Computation: $y = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} x$ ***or*** $\begin{aligned} y_0 &= x_0 + x_1 \\ y_1 &= x_0 - x_1 \end{aligned}$

As graph (direct acyclic graph or DAG):



called a butterfly

http://charlottesmartypants.blogspot.com/2011_02_01_archive.html

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DFT, Size 4

$$\mathbf{DFT}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix}$$

How many (complex) operations to compute the DFT_4 of a (complex) vector?

$$y = \mathbf{DFT}_4 \cdot x$$

12 complex adds/subs and 4 mults by i

Transforms: Examples

A few dozen transforms are relevant

Some examples

$$\mathbf{DFT}_n = [e^{-2k\ell\pi i/n}]_{0\le k,\ell<n}$$

$$\mathbf{RDFT}_n = [r_{k\ell}]_{0\le k,\ell<n}, \quad r_{k\ell} = \begin{cases} \cos\frac{2\pi k\ell}{n}, & k \le \lfloor\frac{n}{2}\rfloor \\ -\sin\frac{2\pi k\ell}{n}, & k > \lfloor\frac{n}{2}\rfloor \end{cases}$$

universal tool

$$\mathbf{DHT} = \left[\cos(2k\ell\pi/n) + \sin(2k\ell\pi/n)\right]_{0\le k,\ell<n}$$

$$\mathbf{WHT}_n = \begin{bmatrix} \mathbf{WHT}_{n/2} & \mathbf{WHT}_{n/2} \\ \mathbf{WHT}_{n/2} & -\mathbf{WHT}_{n/2} \end{bmatrix}, \quad \mathbf{WHT}_2 = \mathbf{DFT}_2$$

$$\mathbf{IMDCT}_n = \left[\cos((2k+1)(2\ell+1+n)\pi/4n)\right]_{0\le k<2n, 0\le\ell<n}$$

MPEG

$$\mathbf{DCT\text{-}2}_n = \left[\cos(k(2\ell+1)\pi/2n)\right]_{0\le k,\ell<n}$$

JPEG

$$\mathbf{DCT\text{-}3}_n = \mathbf{DCT\text{-}2}_n^T \quad \text{(transpose)}$$

$$\mathbf{DCT\text{-}4}_n = \left[\cos((2k+1)(2\ell+1)\pi/4n)\right]_{0\le k,\ell<n}$$

Transform Algorithms

An algorithm for y = Tx is given by a factorization

$$T = T_1 T_2 \cdots T_m$$

Namely, instead of y = Tx we can compute in steps

$$\left.\begin{aligned} t_1 &= T_m x \\ t_2 &= T_{m-1} t_1 \\ &\ldots \\ y &= T_1 t_{m-1} \end{aligned}\right\} \text{ m steps}$$

This reduces the op count only if:
- the T_i are sparse
- m is not too large

Example: Cooley-Tukey Fast Fourier Transform (FFT), size 4

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} x = \begin{bmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & i \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdot & \cdot \\ 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix} x$$

12 adds
4 mults by i

4 adds

1 mult by i

4 adds

0 ops

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Cooley-Tukey FFT, n = 4

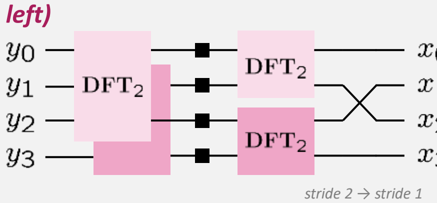
Fast Fourier transform (FFT)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{bmatrix} = \begin{bmatrix} 1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & 1 \\ 1 & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & i \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdot & \cdot \\ 1 & -1 & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 \\ \cdot & \cdot & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

Representation using matrix algebra

$$\mathbf{DFT}_4 = (\mathbf{DFT}_2 \otimes \mathrm{I}_2)\, \mathrm{diag}(1,1,1,i)(\mathrm{I}_2 \otimes \mathbf{DFT}_2)\, \mathrm{L}^4_2$$

Data flow graph (right to left)



2 DFTs of size 2
at stride 2

2 DFTs of size 2

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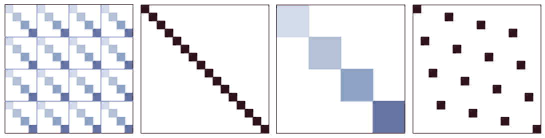
Example Recursive FFT, n = 16, radix 4

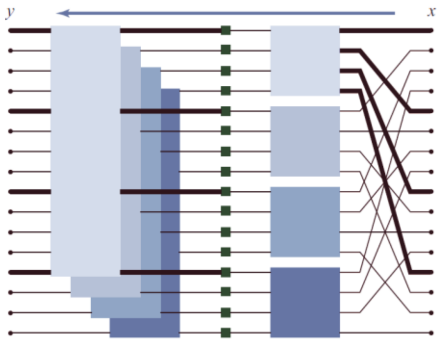
$$\mathrm{DFT}_{16} = \left(\mathrm{DFT}_4 \otimes I_4\right) T_4^{16} \left(I_4 \otimes \mathrm{DFT}_4\right) L_4^{16}$$



Kronecker product:

$A = [a_{ij}]_{i,j}$

$A \otimes B = [a_{ij} \cdot B]_{i,j}$



4 DFTs of size 4 at stride 4

4 DFTs of size 4

General Radix, Recursive Cooley-Tukey FFT

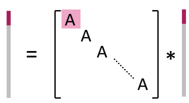
Assume n = km: $\mathbf{DFT}_n = (\mathbf{DFT}_k \otimes \mathrm{I}_m)\, \mathrm{T}_m^n (\mathrm{I}_k \otimes \mathbf{DFT}_m)\, \mathrm{L}_k^n$

permutation matrix

diagonal matrix with roots of unity

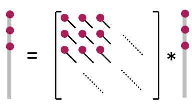
3 key structures: $\mathrm{I}_k \otimes A_m, A_k \otimes \mathrm{I}_m, \mathrm{L}_k^n$

$y = (\mathrm{I}_k \otimes A_m)x$




```
for i = 0:k-1
  y[im:im+m-1] = A*x[im:im+m-1]
```

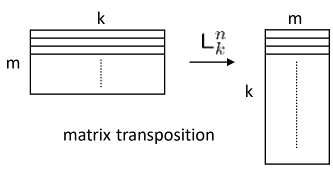
$y = (A_k \otimes \mathrm{I}_m)x$



m A's at stride m

```
for i = 0:m-1
  y[i:m:i+(k-1)m] = A*x[i:m:i+(k-1)m]
```

$y = \mathrm{L}_k^n x$ view x as m x k matrix:



matrix transposition

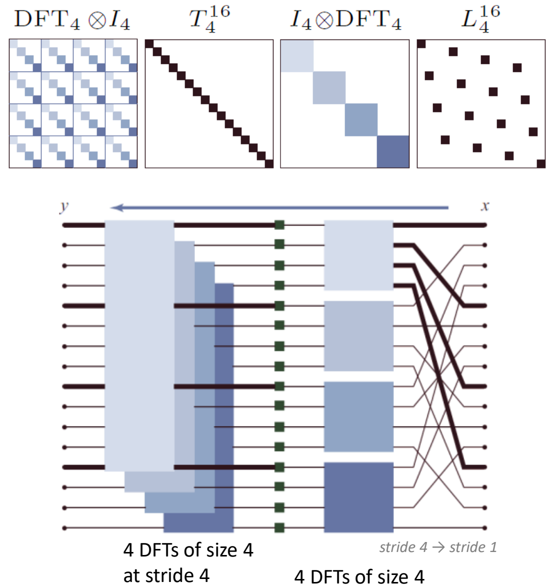
```
for i = 0:k-1
  for j = 0:m-1
    y[im+j] = x[i+kj]
```

read at stride k, write at stride 1

equivalent: read at stride 1, write at stride m

Example FFT, n = 16 *(Recursive, Radix 4)*

$\mathrm{DFT}_{16} =$



DFT_4 is expanded recursively

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Recursive Cooley-Tukey FFT

radix

$$\mathbf{DFT}_{km} = (\mathbf{DFT}_k \otimes \mathrm{I}_m) T^{km}_m (\mathrm{I}_k \otimes \mathbf{DFT}_m) L^{km}_k \qquad \textit{decimation-in-time}$$

$$\mathbf{DFT}_{km} = L^{km}_m (\mathrm{I}_k \otimes \mathbf{DFT}_m) T^{km}_m (\mathbf{DFT}_k \otimes \mathrm{I}_m) \qquad \textit{decimation-in-frequency}$$

For powers of two $n = 2^t$ sufficient together with base case

$$\mathbf{DFT}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Cost:

- *(complex adds, complex mults) = (n $\log_2(n)$, n $\log_2(n)/2$)*
  independent of recursion
- *(real adds, real mults) ≤ (3n $\log_2(n)$, 2n $\log_2(n)$) = 5n $\log_2(n)$ flops*
  depends on recursion: best is at least radix-8

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Recursive vs. Iterative FFT

Recursive, radix-k Cooley-Tukey FFT

$$\mathbf{DFT}_{km} = (\mathbf{DFT}_k \otimes \mathrm{I}_m) T_m^{km} (\mathrm{I}_k \otimes \mathbf{DFT}_m) L_k^{km}$$

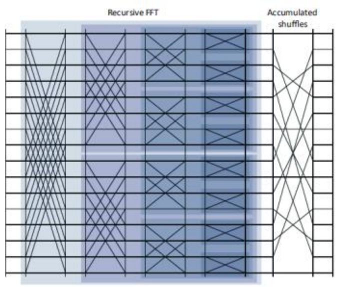
$$\mathbf{DFT}_{km} = L_m^{km} (\mathrm{I}_k \otimes \mathbf{DFT}_m) T_m^{km} (\mathbf{DFT}_k \otimes \mathrm{I}_m)$$

Iterative, radix 2, decimation-in-time/decimation-in-frequency

$$\mathbf{DFT}_{2^t} = \left(\prod_{j=1}^{t} (\mathrm{I}_{2^{j-1}} \otimes \mathbf{DFT}_2 \otimes \mathrm{I}_{2^{t-j}}) \cdot (\mathrm{I}_{2^{j-1}} \otimes T_{2^{t-j}}^{2^{t-j+1}}) \right) \cdot R_{2^t}$$

$$\mathbf{DFT}_{2^t} = R_{2^t} \cdot \left(\prod_{j=1}^{t} (\mathrm{I}_{2^{t-j}} \otimes T_{2^{j-1}}^{2^j}) \cdot (\mathrm{I}_{2^{t-j}} \otimes \mathbf{DFT}_2 \otimes \mathrm{I}_{2^{j-1}}) \right)$$

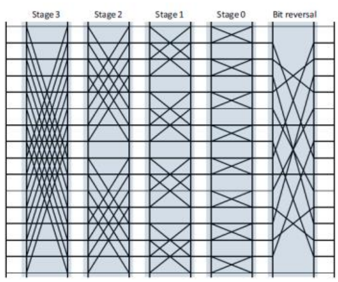
Radix 2, recursive



$$(\mathrm{DFT}_2 \otimes I_8) T_8^{16} \Big(I_2 \otimes \Big((\mathrm{DFT}_2 \otimes I_4) T_4^8 (I_2 \otimes ((\mathrm{DFT}_2 \otimes I_2) T_2^4 (I_2 \otimes \mathrm{DFT}_2) L_2^4)) L_2^8 \Big) \Big) L_2^{16}$$

Radix 2, iterative

same DAG, different execution order



$$\Big((I_1 \otimes \mathrm{DFT}_2 \otimes I_8) D_0^{16} \Big) \Big((I_2 \otimes \mathrm{DFT}_2 \otimes I_4) D_1^{16} \Big) \Big((I_4 \otimes \mathrm{DFT}_2 \otimes I_2) D_2^{16} \Big) \Big((I_8 \otimes \mathrm{DFT}_2 \otimes I_1) D_3^{16} \Big) R_2^{16}$$

Recursive vs. Iterative

Iterative FFT computes in stages of butterflies =
$log_2(n)$ passes through the data

Recursive FFT reduces passes through data =
better locality

Same computation graph but different topological sorting

Rough analogy:

MMM	DFT
Triple loop	Iterative FFT
Blocked	Recursive FFT

The FFT Is Very Malleable

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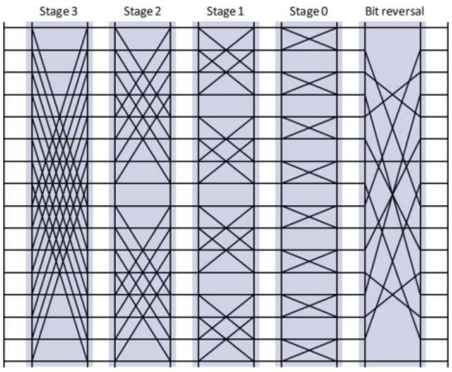
Iterative FFT, Radix 2



$$\left((I_1 \otimes \mathrm{DFT}_2 \otimes I_8) D_0^{16}\right)\left((I_2 \otimes \mathrm{DFT}_2 \otimes I_4) D_1^{16}\right)\left((I_4 \otimes \mathrm{DFT}_2 \otimes I_2) D_2^{16}\right)\left((I_8 \otimes \mathrm{DFT}_2 \otimes I_1) D_3^{16}\right) R_2^{16}$$

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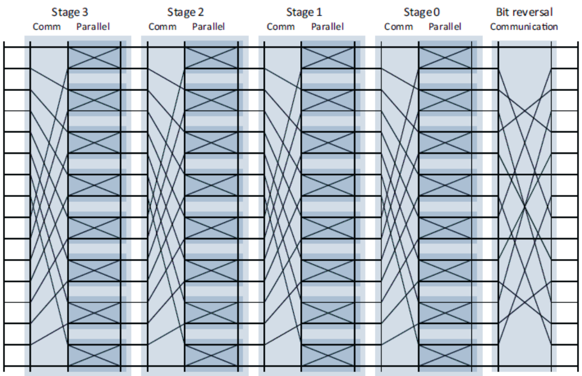
Pease FFT, Radix 2



$$\left(L_2^{16}(I_8 \otimes \mathrm{DFT}_2) D_0^{16}\right)\left(L_2^{16}(I_8 \otimes \mathrm{DFT}_2) D_1^{16}\right)\left(L_2^{16}(I_8 \otimes \mathrm{DFT}_2) D_2^{16}\right)\left(L_2^{16}(I_8 \otimes \mathrm{DFT}_2) D_3^{16}\right) R_2^{16}$$

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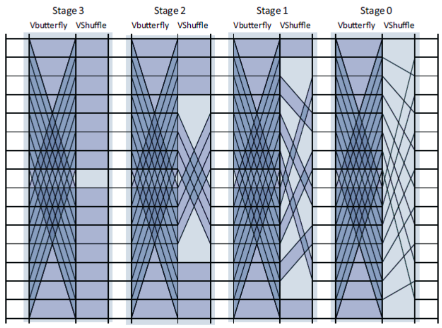
Stockham FFT, Radix 2



$$\Big(\big(\mathrm{DFT}_2\otimes I_8\big)D_0^{16}\big(L_2^2\otimes I_8\big)\Big)\Big(\big(\mathrm{DFT}_2\otimes I_8\big)D_1^{16}\big(L_2^4\otimes I_4\big)\Big)\Big(\big(\mathrm{DFT}_2\otimes I_8\big)D_2^{16}\big(L_2^8\otimes I_2\big)\Big)\Big(\big(\mathrm{DFT}_2\otimes I_8\big)D_3^{16}\big(L_2^{16}\otimes I_1\big)\Big)$$

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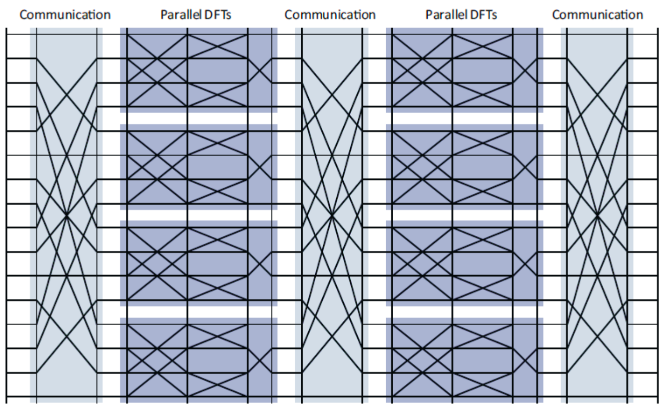
Six-Step FFT



$$L_4^{16}\Big(I_4\otimes\big((\mathrm{DFT}_2\otimes I_2)T_2^4(I_2\otimes \mathrm{DFT}_2)L_2^4\big)\Big)L_4^{16}T_4^{16}\Big(I_4\otimes\big((\mathrm{DFT}_2\otimes I_2)T_2^4(I_2\otimes \mathrm{DFT}_2)L_2^4\big)\Big)L_4^{16}$$

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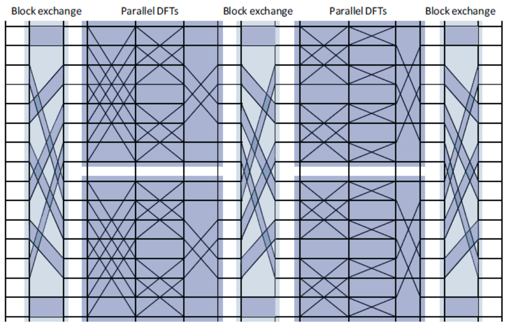
Multi-Core FFT



$$(L_4^8 \otimes I_2)\left(I_2 \otimes \left((\mathrm{DFT}_2 \otimes I_2) T_2^4 (I_2 \otimes \mathrm{DFT}_2) L_2^4\right) \otimes I_2\right)(L_2^8 \otimes I_2)\, T_4^{16} \left(I_2 \otimes \left(I_2 \otimes (\mathrm{DFT}_2 \otimes I_2) T_2^4 (I_2 \otimes \mathrm{DFT}_2)\right) R_2^8\right)(L_2^8 \otimes I_2)$$

Transform Algorithms

$$\mathbf{DFT}_n \to P_{k/2,2m}^{\top}\left(\mathbf{DFT}_{2m} \oplus \left(I_{k/2-1} \otimes_i C_{2m}\,\mathbf{rDFT}_{2m}(i/k)\right)\right)\left(\mathbf{RDFT}'_k \otimes I_m\right), \quad k \text{ even},$$

$$\begin{vmatrix}\mathbf{RDFT}_n\\ \mathbf{RDFT}'_n\\ \mathbf{DHT}_n\\ \mathbf{DHT}'_n\end{vmatrix} \to (P_{k/2,m}^{\top} \otimes I_2)\left(\begin{vmatrix}\mathbf{RDFT}_{2m}\\ \mathbf{RDFT}'_{2m}\\ \mathbf{DHT}_{2m}\\ \mathbf{DHT}'_{2m}\end{vmatrix} \oplus \left(I_{k/2-1} \otimes_i D_{2m} \begin{vmatrix}\mathbf{rDFT}_{2m}(i/k)\\ \mathbf{rDFT}_{2m}(i/k)\\ \mathbf{rDHT}_{2m}(i/k)\\ \mathbf{rDHT}_{2m}(i/k)\end{vmatrix}\right)\right)\left(\begin{vmatrix}\mathbf{RDFT}'_k\\ \mathbf{RDFT}'_k\\ \mathbf{DHT}'_k\\ \mathbf{DHT}'_k\end{vmatrix} \otimes I_m\right), \quad k \text{ even},$$

$$\begin{vmatrix}\mathbf{rDFT}_{2n}(u)\\ \mathbf{rDHT}_{2n}(u)\end{vmatrix} \to L_m^{2n}\left(I_k \otimes_i \begin{vmatrix}\mathbf{rDFT}_{2m}((i+u)/k)\\ \mathbf{rDHT}_{2m}((i+u)/k)\end{vmatrix}\right)\left(\begin{vmatrix}\mathbf{rDFT}_{2k}(u)\\ \mathbf{rDHT}_{2k}(u)\end{vmatrix} \otimes I_m\right),$$

$$\mathbf{RDFT\text{-}3}_n \to (Q_{k/2,m}^{\top} \otimes I_2)\,(I_k \otimes_i \mathbf{rDFT}_{2m})(i+1/2)/k))\,(\mathbf{RDFT\text{-}3}_k \otimes I_m), \quad k \text{ even},$$

$$\mathbf{DCT\text{-}2}_n \to P_{k/2,2m}^{\top}\left(\mathbf{DCT\text{-}2}_{2m} K_2^{2m} \oplus \left(I_{k/2-1} \otimes N_{2m}\,\mathbf{RDFT\text{-}3}_{2m}^{\top}\right)\right) B_n (L_{k/2}^{n/2} \otimes I_2)(I_m \otimes \mathbf{RDFT}'_k) Q_{m/2,k},$$

$$\mathbf{DCT\text{-}3}_n \to \mathbf{DCT\text{-}2}_n^{\top},$$

$$\mathbf{DCT\text{-}4}_n \to Q_{k/2,2m}^{\top}\left(I_{k/2} \otimes N_{2m}\,\mathbf{RDFT\text{-}3}_{2m}^{\top}\right) B'_n (L_{k/2}^{n/2} \otimes I_2)(I_m \otimes \mathbf{RDFT\text{-}3}_k) Q_{m/2,k}.$$

$$\mathbf{DFT}_n \;\to\; (\mathbf{DFT}_k \otimes \mathrm{I}_m)\,\mathrm{T}_m^n (\mathrm{I}_k \otimes \mathbf{DFT}_m)\,\mathrm{L}_k^n, \quad n = km \quad \text{———— \textit{Cooley-Tukey FFT}}$$

$$\mathbf{DFT}_n \;\to\; P_n (\mathbf{DFT}_k \otimes \mathbf{DFT}_m) Q_n, \quad n = km,\ \gcd(k,m) = 1 \quad \text{———— \textit{Prime-factor FFT}}$$

$$\mathbf{DFT}_p \;\to\; R_p^T (\mathrm{I}_1 \oplus \mathbf{DFT}_{p-1}) D_p (\mathrm{I}_1 \oplus \mathbf{DFT}_{p-1}) R_p, \quad p \text{ prime} \quad \text{———— \textit{Rader FFT}}$$

$$\mathbf{DCT\text{-}3}_n \;\to\; (\mathrm{I}_m \oplus \mathrm{J}_m)\,\mathrm{L}_m^n (\mathbf{DCT\text{-}3}_m(1/4) \oplus \mathbf{DCT\text{-}3}_m(3/4)) \cdot (\mathrm{F}_2 \otimes \mathrm{I}_m) \begin{bmatrix}\mathrm{I}_m & 0 \oplus -\mathrm{J}_{m-1}\\ & \frac{1}{\sqrt{2}}(\mathrm{I}_1 \oplus 2\,\mathrm{I}_m)\end{bmatrix}, \quad n = 2m$$

$$\mathbf{DCT\text{-}4}_n \;\to\; S_n \mathbf{DCT\text{-}2}_n\, \mathrm{diag}_{0 \le k < n}(1/(2\cos((2k+1)\pi/4n)))$$

$$\mathbf{IMDCT}_{2m} \;\to\; (\mathrm{J}_m \oplus \mathrm{I}_m \oplus \mathrm{I}_m \oplus \mathrm{J}_m)\left(\left(\begin{bmatrix}1\\-1\end{bmatrix} \otimes \mathrm{I}_m\right) \oplus \left(\begin{bmatrix}-1\\-1\end{bmatrix} \otimes \mathrm{I}_m\right)\right) \mathrm{J}_{2m}\,\mathbf{DCT\text{-}4}_{2m}$$

$$\mathbf{WHT}_{2^k} \;\to\; \prod_{i=1}^{t} (\mathrm{I}_{2^{k_1+\cdots+k_{i-1}}} \otimes \mathbf{WHT}_{2^{k_i}} \otimes \mathrm{I}_{2^{k_{i+1}+\cdots+k_t}}), \quad k = k_1 + \cdots + k_t$$

$$\mathbf{DFT}_2 \;\to\; \mathrm{F}_2$$

$$\mathbf{DCT\text{-}2}_2 \;\to\; \mathrm{diag}(1, 1/\sqrt{2})\,\mathrm{F}_2$$

$$\mathbf{DCT\text{-}4}_2 \;\to\; \mathrm{J}_2\,\mathrm{R}_{13\pi/8}$$

Complexity of the DFT

Measure: L_c, $2 \leq c$

- *Complex adds count 1*
- *Complex mults by a constant a with |a| < c counts 1*
- *L_2 is strictest, L_∞ the loosest (and most natural)*

Upper bounds:

- *$n = 2^k$:* $L_2(DFT_n) \leq 3/2\ n \log_2(n)$ *(using Cooley-Tukey FFT)*
- *General n:* $L_2(DFT_n) \leq 8\ n \log_2(n)$ *(needs Bluestein FFT)*

Lower bound:

- *Theorem by Morgenstern: If $c < \infty$, then $L_c(DFT_n) \geq ½\ n \log_c(n)$*
- *Implies: in the measure L_c for $c < \infty$ the DFT is $\Theta(n \log(n))$*

Lowest Known FFT Cost (Powers of 2)

A modified split-radix FFT with fewer arithmetic operations, *Johnson and Frigo, IEEE Trans. Signal Processing 55(1), pp. 111-119, 2007*

Number of flops ($n = 2^k$):

$$\frac{34}{9} n \log_2(n) - \frac{124}{27} n - 2\log_2(n) - \frac{2}{9}(-1)^{\log_2(n)} \log_2(n) + \frac{16}{27}(-1)^{\log_2(n)} + 8$$

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History of FFTs

The advent of digital signal processing is often attributed to the FFT
(Cooley-Tukey 1965)

History:

- *Around 1805: FFT discovered by Gauss [1]*
  (Fourier publishes the concept of Fourier analysis in 1807!)
- *1965: Rediscovered by Cooley-Tukey*

[1]: Heideman, Johnson, Burrus: "Gauss and the History of the Fast Fourier Transform" Arch. Hist. Sc. 34(3) 1985

Carl-Friedrich Gauss

1777–1855

Contender for the greatest mathematician of all times

Some contributions: Modular arithmetic, least square analysis, normal distribution, fundamental theorem of algebra, Gauss elimination, Gauss quadrature, Gauss-Seidel, non-Euclidean geometry, …

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